Artin's theorem

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1 Introduction

In general, we can use representations of subgroups to obtain representations of a group itself. One natural question to ask at this point is whether every representation can be induced from some subgroup. Unfortunately, this is not the case.

However, we do have some approximation of this fact, given to us by Artin's theorem. Instead of being true of the characters of a group, it is a theorem about virtual characters, which we introduce in section 2.4. It also does not give us that everything is induced, but rather that some integer multiple is a integer combination of induced things. Nevertheless, it's a pretty neat result and sets the stage for the more powerful and useful Brauer's theorem.

In section 2 we will introduce the notion of a virtual character. In section 3 we will state and prove Artin's theorem. And in section 4 we will present a different proof of the second half of Artin's theorem, which is a particularly nice presentation and also adds some specificity to the result, replacing the claim of "finite" with "the order of G".

My treatment in this paper follows [1] chapter 9.

2 The ring of virtual characters

2.1 Representations and characters

Recall the definition of a representation of a finite group G over $\mathbb{C}$ (for simplicity and since we'll only be discussing this case, we assume that G is finite and that the ground field is $\mathbb{C}$, although in general neither of these assumptions need hold to define a representation).

Definition 1. A *representation* of a finite group G over $\mathbb{C}$ is a homomorphism $\rho : G \to GL_n(\mathbb{C})$ for some n.

Given a representation, we can then define the character, which is a function $G \to \mathbb{C}$.

Definition 2. The *character* of a representation $\rho : G \to GL_n(\mathbb{C})$ is given by $\chi_\rho(g) = \mathrm{tr}(\rho(g) : \mathbb{C}^n \to \mathbb{C}^n)$ for $g \in G$.

When the representation ρ is clear we simply write χ without the subscript.

We also define an inner or scaler product on characters, which sometimes proves useful.

Definition 3. Given two characters $\chi_1, \chi_2 : G \to \mathbb{C}$ of representations $\rho_1 : G \to GL(\mathbb{C}^n), \rho_2 : G \to GL(\mathbb{C}^m)$ we define the *scaler product* of χ_1 and χ_2 as $\langle \chi_1, \chi_2 \rangle = \frac{1}{|G|} \sum_G \chi_1(g)\overline{\chi_2(g)}$

Also, remember another central notion of representation theory, that of the *irreducible representation*, which is a representation $\rho : G \to GL(\mathbb{C}^n)$ such that no subspace of $\mathbb{C}^n$ is fixed by all the $\rho(g)$.

Some standard results in representation theory state that for every representation $\rho : G \to GL(\mathbb{C}^n)$ the equation $\rho = \sum_i \langle \chi_\rho, \chi_i \rangle \chi_i$ holds, where the χ_i are all the irreducible representation of G. In particular, $\langle \chi_i, \chi_j \rangle = \delta_{ij}$ for irreducible χ_i, χ_j. Another standard fact is that the number of irreducible representations of a finite group G is also finite. In particular, it is equal to the number of conjugacy classes of G.

2.2 Subgroups

Given a group G and subgroup H there is a pair of adjoint maps which allow us to pass from representations on G to those on H and back. These are the induced representation and the restriction of a representation.

Definition 4. Given a representation on a subgroup $H \subset G$ with character χ there exists an *induced representation* of G with character Ind_H^G is given by $\mathrm{Ind}_H^G(x) = \frac{1}{|H|} \sum_{y \in G : yxy^{-1} \in H} \chi(yxy^{-1})$

The adjoint to this is the restriction of a representation.

Definition 5. Given a representation $\rho : G \to GL(\mathbb{C}^n)$ and subgroup $H \subset G$ we can obtain a representation of H given by $\rho|_H : H \to GL(\mathbb{C}^n)$.

2.3 Class functions

We now introduce the notion of a class function.

Definition 6. A *class function* on a finite group G is a function $f : G \to \mathbb{C}$ such that $f(g) = f(hgh^{-1})$ for all $g, h \in G$.

That is, a class function is a function $G \to \mathbb{C}$ which can be defined on the conjugacy classes of G. It turns out that characters are indeed class functions.

Theorem 7. *A character $\chi : G \to \mathbb{C}$ is a class function.*

Proof. We have $\chi(hgh^{-1}) = \mathrm{tr}(\rho(hgh^{-1})) = \mathrm{tr}(\rho(h)\rho(g)\rho(h^{-1})) = \mathrm{tr}(\rho(h^{-1})\rho(h)\rho(g)) = \mathrm{tr}(\rho(g)) = \chi(g)$. ∎

In fact, the characters of irreducible representations of G provide a basis for the $\mathbb{C}$-vector space of class functions on G, which we call $F_{\mathbb{C}}(G)$. So that we can write $F_{\mathbb{C}}(G) \cong \mathbb{C}\chi_1 \oplus \cdots \oplus \mathbb{C}\chi_n$. We do not prove this fact here, but it is relatively standard, and is demonstrated using an analogue of the scaler product.

We can also define a ring structure on the space of class functions, since if f and h depend only on conjugacy class their product will as well. That is, we define $(fh)(g) = f(g)h(g)$. The distributivity of this multiplication follows from the distributivity of $\mathbb{C}$. The multiplicative identity will be the class function $f(g) = 1$ for all g. Incidentally, this is the character of the irreducible representation known as the trivial representation, given by $\rho(g) = 1$ for all $g \in G$, where 1 is viewed as an element of $GL_1(\mathbb{C}) \cong \mathbb{C}^*$. The other ring axioms follow similarly.

2.4 Virtual characters

The virtual characters form a subring of the ring of class functions, and is what we obtain when we allow characters, their negatives, and sums of these. More precisely:

Definition 8. The subring $R(G) \subset F_{\mathbb{C}}(G)$ of virtual characters is given by integer combinations $\mathbb{Z}\chi_1 \oplus \cdots \oplus \mathbb{Z}\chi_n$.

This is closed under multiplication since the product of two characters $\chi_1 \cdot \chi_2$ is the character of the tensor of the corresponding representations. We state this formally:

Definition 9. Given two representations $\rho_1 : G \to GL(\mathbb{C}^n)$ and $\rho_2 : G \to GL(\mathbb{C}^m)$ we can define the *tensor representation* $\rho_1 \otimes \rho_2 : G \to GL(\mathbb{C}^n \otimes \mathbb{C}^m)$ by $\rho_1 \otimes \rho_2(g) = \rho_1(g) \otimes \rho_2(g)$ and this then has character $\chi_1 \cdot \chi_2(g) = \chi_1(g)\chi_2(g)$ where χ_1 and χ_2 are the characters of ρ_1 and ρ_2 respectively.

The notion of a virtual character arises naturally from the notion of character when we simply include the negative of a character as well.

2.5 Homomorphisms

The maps from section 2.2 define ring homomorphisms. That is, given $H \subset G$ there is a ring homomorphism Ind: $R(H) \to R(G)$ which sends a character to its induced character. And there is an adjoint family of homomorphisms Res: $R(G) \to R(H)$ sending a character to the character of the restriction of the representation.

Theorem 10. *The image of the homomorphism* Ind: $R(H) \to R(G)$ *is an ideal of* $R(G)$.

Proof. This follows from the formula Ind$(\phi \cdot \text{Res}(\psi))$ =Ind$(\phi) \cdot \psi$, so that if ψ is any element of $R(G)$ and χ is in the image of Ind then $\chi \cdot \psi$ is in the image of Ind as well. ∎

We do not prove that Ind$(\phi \cdot \text{Res}(\psi))$ =Ind$(\phi) \cdot \psi$, but this can be checked by direct computation.

3 Artin's theorem

Artin's theorem tell us that any virtual character of G can be expressed as a rational sum of induced characters.

3.1 theorem and proof

Theorem 11. *Let G be a finite group and let X be some collection of subgroups $H \subset G$, then the union of the conjugates of $H \in X$ is equal to all of $G \iff$ the cokernel of Ind: $\oplus_{H \in X} R(H) \to R(G)$ is finite.*

Proof. We prove the theorem one direction at a time:

$\Longrightarrow$ To show this we can simply show that given the assumption that the union of the conjugates of the $H \in X$ gives all of G then we have that the tensored map $\mathbb{Q} \otimes \text{Ind} : \oplus_{H \in X} \mathbb{Q} \otimes R(H) \to \mathbb{Q} \otimes R(G)$ is surjective. Because then for a given character of $R(G)$ we have that we can express some integer multiple of it as an integer sum of things induced by $R(H)$. In particular this is true of the irreducible characters of G which generate $R(G)$ so that then the dimension would be finite. But it is enough to show then that the tensored map $\mathbb{C} \otimes \text{Ind} : \oplus_{H \in X} \mathbb{C} \otimes R(H) \to \mathbb{C} \otimes R(G)$ is surjective, for if the tensor with $\mathbb{Q}$ isn't surjective then the tensor with $\mathbb{C}$ wouldn't be either. But then we can pass to the adjoint and simply require the dual fact that the map $\mathbb{C} \otimes \text{Res} : \mathbb{C} \otimes R(G) \to \oplus_{H \in X} \mathbb{C} \otimes R(H)$ be injective. But this must be the case since if some class function $f : G \to \mathbb{C}$ restricts to 0 on every subgroup $H \in X$ then since every element $g \in G$ is conjugate to some $h \in H$ for some $H \in X$ so that $g = yhy^{-1}$

for some $y \in G$ then we get $f(g) = f(yhy^{-1}) = f(h) = 0$ for all $g \in G$ so that $f = 0$ and $\mathbb{C} \otimes \mathrm{Res}$ is injective as required.

$\Longleftarrow$ For the other direction, we have that $\mathrm{Ind}_H^G(\chi)(g) = \frac{1}{|H|} \sum_{y \in G: ygy^{-1} \in H} \chi(ygy^{-1})$ so that if $\mathrm{Ind}_H^G(\chi)(g) \neq 0$ then there is some $y \in G$ such that $ygy^{-1} \in H$, or rather, $g \in yHy^{-1}$. This then tells us that any character in $\mathrm{Ind}(\oplus_{H \in X} R(H))$ in zero outside of the union of the conjugates of the $H \in X$. But in order for the cokernel of Ind: $\oplus_{H \in X} R(H) \to R(G)$ to be finite we would need that every element of $R(G)$ has some nonzero integer multiple of itself in the image $\mathrm{Ind}(\oplus_{H \in X} R(H))$, in particular this is true of the trivial representation which is not zero for any element of G. So this then tells us that nothing in G is outside of the union of the conjugates of the $H \in X$, or rather that the union of the conjugates of all the $H \in X$ is equal to all of G, as desired.

■

We then have some sense in which every character of a group is induced, although in a fairly weak form.

3.2 Consequences

Worth noting is that we can in Artin's theorem take the collection of subgroups X to be the set of cyclic subgroups of G. We know that every element of a finite group G appears in at least one cyclic subgroup, namely the one which it generates, so this X will satisfy the condition of Artin's theorem. In particular, we obtain:

Corollary 12. *Every character of a finite group G can be expressed as a rational coefficient linear combination of characters induced by cyclic subgroups of G.*

This too is not a very strong result, but it is a more specific claim than the general statement of Artin's theorem. As mentioned in the introduction, Brauer's theorem is a stronger version of Artin's theorem that manages to tell us more. Brauer's theorem is presented in [1] chapter 10, and lies beyond the scope of this paper.

4 Alternative proof

Another version of the first half of our proof gives us a slightly stronger result and uses a somewhat more interesting construction, so we reconstruct it here as well.

4.1 Cyclic class function

To start, we define a new class function for cyclic groups which will be useful to us.

Definition 13. Let A be a cyclic group, then define θ_A is the function taking $x \in A$ to $|A|$ if x generates A, or rather has order $|A|$, and 0 otherwise.

It is easy to see that θ_A is a class function, as conjugate elements have the same order.

4.2 Constant function

One the one hand, we have that the sum of the induced cyclic functions yields a constant function.

Theorem 14. *If G is a finite group then $\sum_{A\subset G} \mathrm{Ind}_A^G(\theta_A)(g) = |G|$ for all $g \in G$, where the sum runs over the cyclic subgroups $A \subset G$.*

Over here Ind is not acting on a representation but rather on a class function, but the formula is still the same, and so we needn't worry.

Proof. First note that every $g \in G$ is a generator for exactly one cyclic subgroup of G, namely the subgroup that it generates.

We have that

$$\mathrm{Ind}_A^G(\theta_A)(x) = \frac{1}{|A|}\sum_{y\in G: yxy^{-1}\in A} \theta_A(yxy^{-1}) = \frac{1}{|A|}\sum_{y\in G:\langle yxy^{-1}\rangle = A} a = \sum_{y\in G:\langle yxy^{-1}\rangle = A} 1.$$

Now since we are considering $\sum_{A\subset G} \mathrm{Ind}_A^G(\theta_A)(g) = |G|$ ranging over all $A \subset G$ cyclic, for any $y \in G$ there is exactly one A which produces a value of 1, so that this sum becomes $\sum_{y\in G} 1 = |G|$, which is the desired result. ■

4.3 Cyclic is virtual

We also have that these cyclic class functions are in fact virtual characters, that is:

Theorem 15. *For a finite cyclic group A, we have that the above defined class function is a virtual character, namely $\theta_A \in R(A)$.*

Proof. We use induction, if $|A| = 1$ then this is the trivial function equal to the trivial character. Otherwise, by theorem 14 we have that $\sum_{B\subset A} \mathrm{Ind}_B^A(\theta_B) = |A|$ with B running over cyclic subgroups of A. If we pull out the case of $B = A$ from the sum, since A is itself cyclic, we will have that $|A| = \theta_A + \sum_{B\subsetneq A} \mathrm{Ind}_B^A(\theta_B)$. But $|A|$ is just $|A|$ times the trivial representation, so that $|A| \in R(A)$, and $\theta_B \in R(B)$ since $|B| \lneq |A|$ and so by induction hypothesis. This then gives that $\mathrm{Ind}_B^A(\theta_B) \in R(A)$ by the homomorphism, so that in fact $\theta_A = |A| - \sum \mathrm{Ind}_B^A(\theta_B) \in R(A)$ as well, as desired. ■

4.4 Proof

We are now ready for an alternate proof of a slightly stronger version of Artin's theorem.

Note that since X satisfying the assumption of theorem 11 has conjugates covering all of G, for any $g \in G$ we have that $\langle g\rangle \subset yHy^{-1}$ for some $y \in G$ so that the characters induced by the cyclic subgroups are a subset of those induced by all of X by the transitive property of inducing class functions. We then have by theorem 15 that the $\theta_A \in R(A)$ so that by theorem 14 we get that the constant function $|G|$ which is $|G|$ times the trivial representation is in the image of $\mathrm{Ind}\oplus_{H\in X} R(H) \to R(G)$. However, since this is an ideal, we have that $|G|\cdot\psi$ is in the image of Ind: $\oplus_{H\in X} R(H) \to R(G)$ for every $\psi \in R(G)$. This gives a slightly more concrete description of how the virtual characters of G are expressed in terms of induced characters.

References

[1] Jean-Pierre Serre, *trans* Leonard L. Scott. *Linear Representations of Finite Groups.* Springer-Verlag, New York, 1977.